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A Test for Differential Flatness by Reduction to Single Input Systems

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Abstract

For nonlinear control systems (p inputs), we present a test for flatness. The method consists of making an initial guess for $p-1$ of the flat outputs, which may involve parameters still to be determined. A choice of functions of time for the $p-1$ outputs reduce the system to one with a single input. For single input systems the problem of flatness has been solved and thus leads to the identification of the last flat output, or to obstructions to flatness under the hypotheses. We demonstrate the method for a coupled rigid body in $\mathbb{R}^2$ and for a single rigid body in $\mathbb{R}^3$.

Keywords

Nonlinear Control, Differential Flatness, Pfaffian System, Feedback Linearization, Cartan Prolongation

AMS Subject Classification: 93C10, 93B29, 58A15, 58A17, 58A10

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1 Introduction

Differential flatness is an important concept in the theory of underdetermined systems of ordinary differential equations. Roughly speaking, a system

$$F^k(t, x^1, \ldots, x^N, \dot{x}^1, \ldots, \dot{x}^N) = 0 \quad k = 1, \ldots, n < N,$$

is differentially flat if there is a smooth 1-1 correspondence between solutions $x(t)$ of the system and arbitrary functions $y(t)$, where $(y^1, \ldots, y^p) \in \mathbb{R}^p$ $(p = N - n)$, of the form

$$\begin{aligned} x(t) &= g(t, y(t), \ldots, y^{(l)}(t)), \\ y(t) &= h(t, x(t), \ldots, x^{(q)}(t)). \end{aligned}$$

Here g, h are smooth maps and l, q are integers. The $y^{(k)}$ is k^{th} derivative of y. The variables y^j are referred to as flat outputs. The special class of systems given by

$$\dot{x}^i = f^i(t, x^1, \ldots, x^n, u^1, \ldots, u^p), \quad i = 1, \ldots, n$$

are more familiar to control theorists and the flat outputs depend on states, inputs and derivatives of inputs

$$y^j = h^j(t, x, u, u^{(1)}, \ldots, u^{(q)}), \quad j = 1, \ldots, p.$$

The term differential flatness was coined and introduced by Fliess et al. They initially used differential algebra as a tool to define and study differential flatness, see [3, 1]. Later, they introduced differential flatness in the setting of Lie Bäcklund mappings on infinite jet spaces, see [2], which also allowed them to define "orbital flatness", a concept more general than differential flatness. See also [6] for a related approach.

Differential flatness was introduced in the framework of exterior differential systems and Cartan prolongations by van Nieuwstadt et al. in [10]. In this paper we shall use the same framework, except that we keep time as a special variable. Hence all transformations are expected to keep time unchanged. We shall not consider orbital flatness and the term "flat" shall stand for differential flatness.

The importance of flatness to control applications lies in the fact that it provides a systematic and relatively simple way to generate solution trajectories between two given states. One uses the maps g, h to transform between original system space (states as well as inputs) and the smaller dimensional flat output space. See [4] for more details.

In the case of single input systems a complete characterization of differential flatness is available, see e.g. [7]. In that case, flatness is the same as feedback linearizability. In the framework of exterior differential systems, checking for flatness reduces to calculating "derived systems" and checking certain rank and integrability conditions. See [10, 8].

For multi-input systems no complete theory exists. We shall describe a method that involves making a guess for all but one flat output and then reducing the system to a single input case by setting the flat outputs (all but one) to arbitrary functions $f^j(t)$. The flatness of the reduced system can be verified since it corresponds to a single input case.

The paper is organized as follows. Section 2 introduces the notions of Cartan prolongations and absolute equivalence and gives a definition of differential flatness. This section also summarizes relevant existing results. Section 3 describes, with the aid of an example, the method of reducing a multi-input system to a single input one by making a guess for all but one of the flat outputs. This section also gives formal geometric statements and proofs validating the idea behind this approach. Finally Section 4 gives two examples of rigid body systems where this method is applied to test their flatness. The reader interested in a quick overview of the method can skip Section 2 and directly read Section 3.

2 Cartan Prolongations, Absolute Equivalence and Differential Flatness

In this section we develop a mathematical framework for flatness, roughly following [8, 10]. Throughout this paper, time, t, denotes the standard coordinate on $\mathbb{R}$. Maps between manifolds and objects such as forms, vector fields etc. on manifolds are assumed to be C^∞-smooth and submanifolds are assumed to be regular.

Definition 1 A **system** is a triple (M, π, I), where M is a manifold, $\pi : M \to \mathbb{R}$ a submersion and I a finitely generated module of 1-forms on M (i.e. Pfaffian system). For $q \in M$, the **codimension at** q of the system is $\dim M - \dim I(p)$. A system is **trivial** if $I = \{0\}$.

The map0 π selects the time coordinate for the system. Note that $\pi(M)$ is an open interval of $\mathbb{R}$ since submersions are open maps and M is connected.

Definition 2 Let $S = (M, \pi, I)$ be a system. A **solution** of S is a curve $c : (a, b) \to M$ such that $\pi \circ c = \mathrm{id}$ and $c^*(I) = \{0\}$.

It follows that c is an immersion and that the image of the solution, $c(a, b)$, is a submanifold of M.

To see the connection with systems of differential equations, suppose (M, π, I) is a system. Let us consider a coordinate system $(t, x^1, \ldots, x^N)$ on an open set $U \subset M$ (frequently, we shall simply write t for $\pi^* t$.) Suppose that $\{\omega^1, \ldots, \omega^n\}$ $(n < N)$ is a set of linearly independent generators of I and in local coordinates let

$$\omega^i = \sum_{j=1}^{N} a_{i,j}(t, x) dx^j + a_{i,0}(t, x) dt, \quad i = 1, \ldots, n. \tag{1}$$

A solution is given by functions $(x^1(t), \ldots, x^N(t))$ that satisfy the following underdetermined system of differential equations $(n < N)$:

$$\sum_{j=1}^{N} a_{i,j}(t, x) \dot{x}^j + a_{i,0}(t, x) = 0, \quad i = 1, \ldots, n. \tag{2}$$

In general, this system cannot be put in the familiar form $\dot{x} = f(t, x, u)$ by a coordinate change. However, $\{I, dt\}$ is integrable if and only if, in suitable local coordinates $(t, x^1, \ldots, x^n, u^1, \ldots, u^p)$, equation (2) take the form $\dot{x} = f(t, x, u)$ for a control system with p inputs, see [8]. The system has codimension $N - n + 1 = p + 1$.

Definition 3 Let $S_1 = (M_1, \pi_1, I_1)$ and $S_2 = (M_2, \pi_2, I_2)$ be two systems. A **morphism** from S_1 to S_2 is a surjective submersion $\phi : M_1 \to M_2$ with the following properties.

1. $\pi_2 \circ \phi = \pi_1$,
2. $\phi^*(I_2) \subset I_1$.

For a curve c_2 in M_2, a **lift** of c_2 is a curve c_1 in M_1 such that $c_2 = \phi \circ c_1$.

It follows that for a morphism ϕ, each solution of S_1 is the lift of a unique solution of S_2. On the other hand, a solution of S_2 may have 0, 1 or many lifts to a solution of S_1, depending on the solution and the morphism ϕ.

Definition 4 Let ϕ be a morphism from $S_J = (B, \pi_B, J)$ to $S_I = (M, \pi_M, I)$. Then S_J is a **Cartan prolongation** of S_I **via ϕ** if every solution c of S_I has a unique lift to a solution $\tilde{c}$ of S_J. We say S_J is a **Cartan prolongation** of S_I if there exists a morphism ϕ from S_J to S_I such that S_J is a Cartan prolongation of S_I via ϕ.

Definition 5 Two systems (M_1, π_1, I_1) and (M_2, π_2, I_2) are **equivalent** if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\pi_1 = \pi_2 \circ \phi$ and $\phi^*(I_2) = I_1$.

Definition 6 Two systems (M_1, π_{M_1}, I_1) and (M_2, π_{M_2}, I_2) are **absolutely equivalent** if there exist respective Cartan prolongations (B_1, π_{B_1}, J_1) and (B_2, π_{B_2}, J_2) that are equivalent.

We are now ready to give a definition of differential flatness. For a more detailed discussion and its connections with the differential algebraic definition we refer to [10].

Definition 7 A system (M, π_M, I) that is absolutely equivalent to a trivial system $(N, \pi_N, \{0\})$ is **differentially flat** (or simply **flat**). If $(t, y^1, \ldots, y^p)$ are local coordinates on N then $(y^1, \ldots, y^p)$ are a set of **flat outputs**.

In local coordinates $(t, x^1, \ldots, x^N)$ on M and local coordinates $(t, y^1, \ldots, y^p)$ on N it follows, see [10], that the 1-1 correspondence between solutions is, on an open dense set, given by equations of the form

$$\begin{aligned} x(t) &= g(t, y(t), y^{(1)}(t), \ldots, y^{(l)}(t)), \\ y(t) &= h(t, x(t), x^{(1)}(t), \ldots, x^{(q)}(t)). \end{aligned} \tag{3}$$

The number of flat outputs is p where $p+1$ is the codimension of system (M, π_M, I). If the system is a control system then p is also the number of inputs. This can easily be proven in the differential algebraic setting, but to prove in the language of Cartan prolongations one should show that a Cartan prolongation of a system has the same codimension as the system. The authors are not aware of a proof. Also matters are more complicated when systems don't have constant codimension.

It must be observed that in the particular instance when $q = 0$, i.e. the flat outputs only depend on (t, x), the original system is a Cartan prolongation of the trivial system $(N, \pi_N, \{0\})$.

Before we state a familiar result characterizing flat systems of codimension 2, we need the notion of derived system.

Definition 8 Let I be a Pfaffian system on a manifold M and denote by $\Omega^1(M)$ the system of all 1-forms on M. The **derived systems** of I are $I^{(0)} = I$ and, for each $k \geq 0$,

$$I^{(k+1)} = \{\omega \in I^{(k)} \ : \ d\omega = \sum_i \alpha_i \wedge \beta_i, \ \alpha_i \in I^{(k)}, \ \beta_i \in \Omega^1(M)\}.$$

Calculating derived system only involves linear algebra and poses no problems for concrete examples. For the following result we refer to [7, 8].

Theorem 9 A system (M, π_M, I) of constant codimension 2 is flat if and only if

1. $\dim I^{(i)} = \dim I^{(i-1)} - 1$, for $i = 0, \dots, n = \dim I$. This implies $I^{(n)} = \{0\}$.

2. The system $\{I^{(i)}, dt\}$ is integrable for each $i = 0, \dots, n$.

3 Reduction of Higher Codimension Systems to Codimension 2

Theorem 9 characterizes flatness for systems with codimension 2. Although some verifiable necessary [9, 5] and sufficient [10] conditions are known, no complete characterization exists for systems with higher codimension. Deciding whether such a system is flat involves making an educated guess based on the special structure of the system and experience.

We will now describe a method that determines whether a system has flat outputs of a particular form. We will only look for flat outputs that depend on the original variables (t, x) of the system and not on the derivatives of x. In other words, we will only check if the given system is a Cartan prolongation of a trivial system. This may seem restrictive, but it is not. In fact, if we suspect that the flat outputs depend on up to q derivatives of x then we first prolong the given system by differentiation q times and then take the resulting system as our starting point.

Assume we have a system with p inputs. The first step of the method involves making a guess for $p-1$ flat outputs $y^1, \dots, y^{p-1}$. Often, this guess will involve expressing the flat outputs as a parameterized family. A simple example will serve to illustrate the idea. Consider the system of differential equations

$$\begin{aligned} x^2\dot{x}^1 - x^1\dot{x}^2 &= x^3, \\ x^1\dot{x}^3 &= x^4, \end{aligned} \tag{4}$$

corresponding to a system $(\mathbb{R}^5, \pi, I)$, where $\pi(t, x) = t$ and $I = \{x^2dx^1 - x^1dx^2 - x^3dt, x^1dx^3 - x^4dt\}$. We "guess" that one of the flat outputs is given by $y = x^1 - \lambda x^4$, where λ is constant.

The second step in the method specifies the outputs to free functions of time: $y^i = Y^i(t), i = 1, \ldots, p-1$. Solve for (some of) the variables x in terms of the free functions $Y(t)$, and substitute them in the system equations. This leads to a system, for which Theorem 9 applies. Note that the resulting system is often time dependent. For the example, set $y = Y(t)$ for arbitrary $Y : \mathbb{R} \to \mathbb{R}$. Then $x^1 = Y(t) + \lambda x^4$, and substituting this into (4) yields,

$$\begin{aligned} \lambda x^2 \dot{x}^4 + x^2 \dot{Y}(t) - (Y(t) + \lambda x^4)\dot{x}^2 &= x^3, \\ (\lambda x^4 + Y(t))\dot{x}^3 &= x^4. \end{aligned} \tag{5}$$

This system, which we call the reduced system, is underdetermined by 1 equation as opposed to 2 in the case of the original system.

The third step of the method checks whether the conditions of Theorem 9 are satisfied. In the case that they are, a flat output z for the reduced system can be calculated. In general, this flat output will depend on (t, x) and the free functions $Y^i(t)$, but in order that z is the final flat output for the original system, it is necessary that $z = h(t, x)$. For the example, the Pfaffian system of the reduced (restricted) system (5) is given by

$$\begin{aligned} \bar{I} = \{ &\lambda x^2 dx^4 - (Y(t) + \lambda x^4) dx^2 - (x^3 - x^2 Y'(t)) dt, \\ &(Y(t) + \lambda x^4) dx^3 - x^4 dt \}. \end{aligned} \tag{6}$$

Calculations show that $\{\bar{I}, dt\}$ is integrable and $\bar{I}^{(1)}$ drops rank by one i.e. $\dim \bar{I}^{(1)} = \dim \bar{I} - 1$. In fact, $\bar{I}^{(1)} = \{\alpha\}$, where

$$\begin{aligned} \alpha = &-Y(t)^2 dx^2 - \lambda Y(t) x^4 dx^2 + 2\lambda x^3 Y(t) dx^3 + 2\lambda^2 x^3 x^4 dx^3 \\ &+ \lambda x^2 Y(t) dx^4 - (2\lambda x^3 x^4 + Y(t) x^3 - Y(t) Y'(t) x^2) dt \}. \end{aligned} \tag{7}$$

Since $d\alpha = 2\lambda^2 x^3 dx^3 \wedge dx^4$ modulo α, dt, $\lambda = 0$ is the only value for which $\{\bar{I}^{(1)}, dt\}$ is integrable. For this choice $\{\bar{I}^{(1)}, dt\}$ is integrable and $\bar{I}^{(2)} = \{0\}$. Moreover, $\alpha = Y(t)(-Y(t)dx^2 + (x^2 Y'(t) - x^3)dt)$, indicates that x_2 is a flat output for the reduced system. We have thus found x^1 and x^2 to be a set of flat outputs.

In order to see the geometric meaning of this method, suppose we start with a system S. We are interested in knowing if S is a Cartan prolongation of some trivial system S_1 with a corresponding morphism ϕ_1. We consider a trivial system S_2 that corresponds to the subset of all but one flat output that we have guessed. There is a morphism ϕ_2 from S to S_2 that relates the flat outputs as functions of coordinates (t, x) of S. Thus in the above example,

$S_2 = (\mathbb{R}^2, \pi_2, \{0\})$ and in local coordinates (t, y) on $\mathbb{R}^2$, $\phi_2 : (t, x) \to (t, y = x^1 - \lambda x^4)$. If our guess is correct, then there must be a morphism $\phi_{1,2}$ from S_1 to S_2 which just picks out the subset of flat outputs. This means, $\phi_2 = \phi_{1,2} \circ \phi_1$.

Having decided on S_2 and ϕ_2 our method involves choosing an arbitrary solution c of S_2 and looking at the restriction of S to the fibers of ϕ_2 over the image of c. The Proposition 11 (which is more general in that S_1 and S_2 need not be trivial systems) asserts the validity of our approach. Also since codimension of S_2 is one less than that of S and S_1, the restriction $S|_{\phi_2^{-1} \circ c(a,b)}$ has codimension 2. By Theorem 9, it may then be verified whether S is a Cartan prolongation of some trivial system.

Definition 10 Let $S = (M, \pi, I)$ be a system and suppose $N \subset M$ is a submanifold of M such that $\pi|_N : N \to \mathbb{R}$ is a submersion. Then the **restriction of S to N** is $S|_N = (N, \pi|_N, i_N^*(I))$, where i_N is the inclusion $N \to M$.

Proposition 11 Let $S = (M, \pi_M, I)$ and $S_i = (M_i, \pi_{M_i}, I_i), i = 1, 2$, be systems. Let ϕ_1 and ϕ_2 be morphisms from S to S_1 and S_2 respectively. Furthermore suppose $\phi_{1,2}$ is a morphism from S_1 to S_2 as in the diagram

$$\begin{tikzcd} S \arrow[rr, "\phi_1"] \arrow[dr, "\phi_2"'] & & S_1 \arrow[dl, "\phi_{1,2}"] \\ & S_2 & \end{tikzcd}$$

and $\phi_{1,2} \circ \phi_1 = \phi_2$.

Then, S is a Cartan prolongation of S_1 if and only if for every solution $c : (a, b) \to M_2$ of S_2, $S|_{\phi_2^{-1} \circ c(a,b)}$ is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ via $\phi_1|_{\phi_2^{-1} \circ c(a,b)}$.

The proof reduces to a series of lemmas.

Lemma 12 Let $S = (M, \pi, I)$ be a system and $S|_N = (N, \pi|_N, i_N^*(I))$ be its restriction. Let $c : (a, b) \to N$ be a curve such that $\pi \circ c = \text{id}$. Then c is a solution of S if and only if it is a solution of $S|_N$.

Proof Follows from $(i_N \circ c)^*(I) = c^*(i_N^* I)$. ∎

Lemma 13 Let ϕ be a morphism from S_1 to S_2. Let $S_2|_N$ be a restriction of S_2. Then $S_1|_{\phi^{-1}(N)}$ is a well defined restriction of S_1 and $\phi|_{\phi^{-1}(N)}$ is a morphism from $S_1|_{\phi^{-1}(N)}$ to $S_2|_N$.

Proof Since ϕ is a submersion, $\phi^{-1}(N)$ is a submanifold and $\phi|_{\phi^{-1}(N)}$ is a surjective submersion onto N. ∎

Remark 14 We need to make sure that the restricted systems in the statement of the proposition are well defined. First note that $S_2|_{c(a,b)}$ is a well defined restriction and hence from the Lemma 13 it follows that $S_1|_{\phi_{1,2}^{-1}\circ c(a,b)}$ and $S|_{\phi_2^{-1}\circ c(a,b)}$ are well defined and $\phi_1|_{\phi_2^{-1}\circ c(a,b)}$ is a morphism from $S|_{\phi_2^{-1}\circ c(a,b)}$ to $S_1|_{\phi_{1,2}^{-1}\circ c(a,b)}$.

Lemma 15 Let S_1 be a Cartan prolongation of S_2 via ϕ. Let $S_2|_N$ be a restriction of S_2. Then $S_1|_{\phi^{-1}(N)}$ is a Cartan prolongation of $S_2|_N$ via $\phi|_{\phi^{-1}(N)}$.

Proof Follows from Lemmas 12 and 13. ∎

Proof (of Proposition 11) *only if:* This follows from Lemma 15.
if: Let c_1 be a solution of S_1. First we show it has a lift. Let $c = \phi_{1,2}\circ c_1$. Then c is a solution of S_2 and hence $S|_{\phi_2^{-1}\circ c(a,b)}$ is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1}\circ c(a,b)}$ by assumption. But, by Lemma 12 above, c_1 is a solution of $S_1|_{\phi_{1,2}^{-1}\circ c(a,b)}$ and hence there is a unique lift $\tilde{c}$, which is a solution of $S|_{\phi_2^{-1}\circ c(a,b)}$ and hence a solution of S appealing again to Lemma 12. To show uniqueness of lift, suppose $\tilde{c_2}$ is another lift of c_1. Then it follows that $\tilde{c_2}$ is also a solution of $S|_{\phi_2^{-1}\circ c(a,b)}$, violating the unique lift of $S|_{\phi_2^{-1}\circ c(a,b)}$ being a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1}\circ c(a,b)}$. ∎

So far, we have only discussed the scenario where the test succeeds for some parameter value. The test fails if the reduced system is not flat. To illustrate this, consider the same example and suppose we choose $\lambda = 1$, i.e. we guess that $x^1 - x^4$ is a flat output. Our calculations show that the reduced system cannot be flat for any choice of $Y(t)$. Hence Proposition 11 tells us that the system cannot be flat with $x^1 - x^4$ and a function of $(t, x^1, \dots, x^4)$ as the flat outputs. However, one may wonder whether there exists a function of t, x^i and finitely many derivatives of x^i that together with $x^1 - x^4$ forms a set of flat outputs. The following proposition shows that this is not possible.

Proposition 16 Let $S = (M, \pi_M, I)$ and $S_i = (M_i, \pi_{M_i}, I_i), i = 1, 2$, be systems. Let ϕ_2 be a morphism from S to S_2 and $\phi_{1,2}$ be a morphism from S_1 to S_2, as in the diagram,

$$\begin{tikzcd}
& \tilde{S} \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\phi}_1"] & \\
S \arrow[dr, "\phi_2"'] & & S_1 \arrow[dl, "\phi_{1,2}"] \\
& S_2 &
\end{tikzcd}$$

and suppose S is absolutely equivalent to S_1. Then, for every solution $c : (a, b) \to M_2$ of S_2, $S|_{\phi_2^{-1} \circ c(a,b)}$ and $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$ are absolutely equivalent.

Proof Since S and S_1 are absolutely equivalent there exists a system $\tilde{S} = (B, \pi_B, J)$ which is a Cartan prolongation of S and S_1. Let $\tilde{\phi} : B \to M$ and $\tilde{\phi}_1 : B \to M_1$ be the corresponding morphisms. Then by the Lemmas 13 and 15 $\tilde{S}|_{\tilde{\phi}^{-1} \circ \phi_2^{-1} \circ c(a,b)}$ is a valid restriction and is a Cartan prolongation of $S|_{\phi_2^{-1} \circ c(a,b)}$. But it is also the same as $\tilde{S}|_{\tilde{\phi}_2^{-1} \circ \phi_{1,2}^{-1} \circ c(a,b)}$ and is a Cartan prolongation of $S_1|_{\phi_{1,2}^{-1} \circ c(a,b)}$. ∎

When S_1 and S_2 are trivial and S_2 has codimension one less than S_1 the situation corresponds to our method. Then the proposition says that S is flat with M_2 providing all but one flat output, only if every restriction $S|_{\phi_2^{-1} \circ c(a,b)}$ is flat. In our example, when $\lambda \neq 0$ the reduced system (restriction) fails to be flat and hence $x^1 - x^4$ cannot be a flat output.

4 Examples

Recently, various mechanical systems have been found to be flat with coordinates of a body fixed point providing a subset of the flat outputs; see [4] for some examples. With the method developed in the previous section, one may systematically search for such flat outputs. We demonstrate this for two mechanical control systems. It was a surprise to the authors to discover that the first example is flat.

4.1 Planar Coupled Rigid Bodies with 3 Inputs

The system we consider consists of 2 planar rigid bodies hinged at a point, O, and moving under gravity, g (see Figure 1). Two of the inputs, f_1, f_2, are body fixed forces acting on the first body such that their lines of action intersect at a point P on the line joining the point O and G_1, the center of mass of the first body. The third input is a pure torque, τ, between the two bodies, i.e. equal and opposite torques on the two bodies. Let $OG_i = r_i$, $\sigma_i = \sqrt{J_i/m_i}$ where J_i and m_i are the moment of inertia and the mass of body i. Furthermore, assume $OP = 1$, the mass of the first body $m_1 = 1$, and $m_2 = \mu$. From a Lagrangian point of view, the system evolves on the configuration manifold $\mathbb{R}^2 \times S^1 \times S^1$, with coordinates $(x, y, \theta_1, \theta_2)$. The equations of motion are given by

$$
\begin{aligned}
&(1+\mu)(\ddot{x}\sin\theta_1 - \ddot{y}\cos\theta_1 - g\cos\theta_1) + r_1\dot{\theta}_1^{\,2} \\
&\qquad + \mu r_2 \sin(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\cos(\theta_2-\theta_1)\dot{\theta}_2^{\,2} = f_1 \\
&(1+\mu)(\ddot{x}\cos\theta_1 + \ddot{y}\sin\theta_1 + g\sin\theta_1) - r_1\ddot{\theta}_1 \\
&\qquad - \mu r_2\cos(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\sin(\theta_2-\theta_1)\dot{\theta}_2^{\,2} = -f_2 \\
&(r_1^2+\sigma_1^2)\ddot{\theta}_1 - r_1\cos\theta_1\ddot{x} - r_1\sin\theta_1\ddot{y} - g r_1\sin\theta_1 = f_2 + \tau \\
&(r_2^2+\sigma_2^2)\ddot{\theta}_2 - r_2\cos\theta_2\ddot{x} - r_2\sin\theta_2\ddot{y} - g r_2\sin\theta_2 = -\tau.
\end{aligned}
\tag{8}
$$

The system can be written as a Pfaffian system of codimension 4. The single second order equation, obtained by eliminating f_2 and τ from the last three equations, corresponds to a Pfaffian system of codimension 4 in coordinates $(t, x, y, \theta_1, \theta_2, \dot{x}, \dot{y}, \dot{\theta}_1, \dot{\theta}_2)$. The full system is a Cartan prolongation of this latter system, because, given any solution of the latter, there is a unique corresponding solution for the full system, in which $(x, y, \theta_1, \theta_2, \dot{x}, \dot{y}, \dot{\theta}_1, \dot{\theta}_2)$ are the same, and (f_1, f_2, τ) are given by above equations. We look for flat outputs that only depend on configuration and velocity variables. In other words, check whether the simpler system is a Cartan prolongation of a trivial system. Our starting point is the following differential equation.

$$
\begin{aligned}
&(r_1^2+\sigma_1^2)\ddot{\theta}_1 - r_1\cos\theta_1\ddot{x} - r_1\sin\theta_1\ddot{y} - g r_1\sin\theta_1 \\
&\qquad + (r_2^2+\sigma_2^2)\ddot{\theta}_2 - r_2\cos\theta_2\ddot{x} - r_2\sin\theta_2\ddot{y} - g r_2\sin\theta_2 \\
&\qquad\quad + (1+\mu)(\ddot{x}\cos\theta_1 + \ddot{y}\sin\theta_1 + g\sin\theta_1) - r_1\ddot{\theta}_1 \\
&\qquad\qquad - \mu r_2\cos(\theta_2-\theta_1)\ddot{\theta}_2 + \mu r_2\sin(\theta_2-\theta_1)\dot{\theta}_2^{\,2} = 0.
\end{aligned}
$$

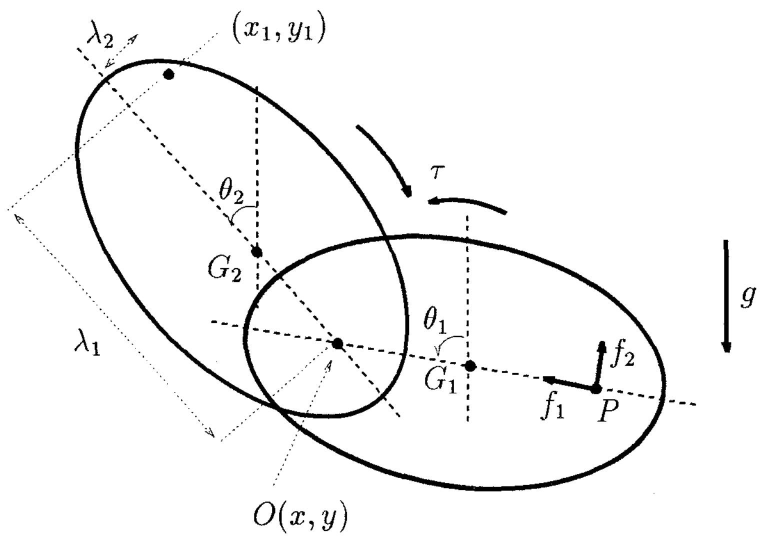

Figure 1: Two coupled rigid bodies in $\mathbb{R}^2$

This corresponds to a Pfaffian system,

$$
\begin{aligned}
\{d\theta_1 - \dot{\theta}_1 dt, d\theta_1 - \dot{\theta}_1 dt,& \\
&(r_1^2 + \sigma_1^2)d\dot{\theta}_1 - r_1 \cos\theta_1 d\dot{x} - r_1 \sin\theta_1 d\dot{y} - g r_1 \sin\theta_1 dt \\
&+ (r_2^2 + \sigma_2^2)d\dot{\theta}_2 - r_2 \cos\theta_2 d\dot{x} - r_2 \sin\theta_2 d\dot{y} - g r_2 \sin\theta_2 dt \\
&\quad + (1+\mu)(d\dot{x}\cos\theta_1 + d\dot{y}\sin\theta_1 + g\sin\theta_1)dt - r_1 d\dot{\theta}_1 \\
&\qquad - \mu r_2 \cos(\theta_2 - \theta_1)d\dot{\theta}_2 + \mu r_2 \sin(\theta_2 - \theta_1)\dot{\theta}_2^{\,2} dt\}.
\end{aligned}
$$

We are looking for 3 flat outputs and to use the method we need to guess some form for 2 of them. We test if the system is flat with 2 of the flat outputs given by coordinates of a body fixed point in the second body (intuitively, the second body is a more reasonable guess than the first body where the forces are applied.)

Coordinates (x_1, y_1) of a body fixed point are given by,

$$
\begin{aligned}
x_1 &= x - \lambda_1 \sin\theta_2 + \lambda_2 \cos\theta_2, \\
y_1 &= y + \lambda_1 \cos\theta_2 + \lambda_2 \sin\theta_2,
\end{aligned}
\tag{10}
$$

where (λ_1, λ_2) are its coordinates in the body fixed frame and as such are constants. Restricting the system to $x_1 = X_1(t)$ and $y_1 = Y_1(t)$ for arbitrary

$X_1, Y_1 : \mathbb{R} \to \mathbb{R}$ corresponds to substituting

$$\begin{aligned} x &= X_1(t) + \lambda_1 \sin\theta_2 - \lambda_2 \cos\theta_2, \\ y &= Y_1(t) - \lambda_1 \cos\theta_2 - \lambda_2 \sin\theta_2 \end{aligned} \tag{11}$$

in the Pfaffian system. We get a codimension 2 system I that has three forms in coordinates $(t, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$. Calculations reveal that the derived system $I^{(1)}$ drops rank by 1, $(\dim I^{(1)} = \dim I - 1)$, and $\{I, dt\}$ is integrable. Further calculations show that $I^{(2)}$ drops rank by one only if certain algebraic relations amongst system parameters (μ, r_i, σ_i) and λ_1, λ_2 hold. For generic parameter values these relations are:

$$\lambda_1 = \frac{\mu r_2}{1 + \mu - r_1}, \quad \lambda_2 = 0. \tag{12}$$

For this choice all the necessary and sufficient conditions for flatness are satisfied. In other words, $I^{(2)}$ and $I^{(3)}$ drop rank by one and each $\{I^{(i)}, dt\}$ is integrable. Also a flat output can be obtained from the form $dz - wdt$ that generates $I^{(2)}$. It is given by

$$\begin{aligned} z = (&-r_1^3 + 2r_1^2 - r_1 + \sigma_1^2\mu + r_1^2\mu - \sigma_1^2 r_1 - r_1\mu + \sigma_1^2)\theta_1 \\ &+(-\sigma_2^2 r_1 + \sigma_2^2\mu - r_2^2 r_1 + \sigma_2^2 + r_2^2)\theta_2. \end{aligned} \tag{13}$$

Note that z is well-defined on the manifold of the original system. In particular, z does not depend on $f(t)$ or $h(t)$. Therefore, the original system is indeed flat.

Observe that x, y, θ_1 can be solved in terms of the flat outputs and θ_2. Substituting these in (9), we obtain an algebraic equation involving the flat outputs, their derivatives and θ_2. So θ_2 can be solved from this equation in terms of the flat outputs and their derivatives. The solution, however, may not be unique, but the set of solutions is discrete.

Remark 17 In practice one often has to let go of the unique lifting condition for Cartan prolongations, and accept cases that only have locally (in the fiber) a unique lift. From the point of view of applications this is not a problem.

So we conclude that the system is differentially flat. Two of the flat ouputs are given by the body fixed point located on the line OG_2 and a distance $\mu r_2/(1 + \mu - r_1)$ from O. The third output is a linear combination of the angles, as given in equation (13).

4.2 3D Rigid Body with 4 Inputs

Consider a 3D-rigid body without gravity acted upon by 4 inputs, 3 of which are forces acting through a point P fixed in the body which is different from G the center of mass, see Figure 2. We choose a body fixed orthonormal frame of reference (e_1, e_2, e_3), which coincides with the principal axes of moment of inertia and assume further that PG coincides with the direction of e_1. The fourth input is a torque about e_1. Let (E_1, E_2, E_3) be an orthonormal stationary frame. Let R be the rotation matrix with columns R_1, R_2, R_3 which correspond to coordinate vectors of e_1, e_2, e_3 with respect to the basis (E_1, E_2, E_3).

Let $x \in \mathbb{R}^3$ be the position vector of G in the stationary frame, Let I_i the principal moment of inertia in the e_i direction, M the mass of the body and $PG = l$. Let $\omega \in \mathbb{R}^3$ be the angular velocity of the body relative to the stationary frame but expressed with respect to the basis (e_i). And finally, let T be the torque about e_1 and F_i be the force in e_i direction.

The dynamics of the system are described by:

$$\begin{aligned}
T &= I_1\dot{\omega_1} + (I_3 - I_2)\omega_2\omega_3, \\
F_3 l &= I_2\dot{\omega_2} + (I_1 - I_3)\omega_3\omega_1, \\
-F_2 l &= I_3\dot{\omega_3} + (I_2 - I_1)\omega_1\omega_2, \\
F_1 &= MR_1^T\ddot{x}, \\
F_2 &= MR_2^T\ddot{x}, \\
F_3 &= MR_3^T\ddot{x}.
\end{aligned}$$

The vector ω is related to R by $\omega^\wedge = R^T\dot{R}$; the operator $^\wedge$ maps a vector $u \in \mathbb{R}^3$ to a 3×3 skew symmetric matrix $u^\wedge$ that satisfies, $u \times y = u^\wedge y$, for every $y \in \mathbb{R}^3$, where $\times$ is the familiar cross product of vectors in $\mathbb{R}^3$.

This is a system on the manifold $\mathbb{R} \times TSE(3) \times \mathbb{R}^3$. Just as in the previous example, a Cartan prolongation can be stripped off by eliminating F_2 and F_3 and discarding the equations involving T and F_1. We get the following two equations:

$$\begin{aligned}
-R_2^T\ddot{x} &= k_3\dot{\omega_3} + (k_2 - k_1)\omega_1\omega_2, \\
R_3^T\ddot{x} &= k_2\dot{\omega_2} + (k_1 - k_3)\omega_3\omega_1,
\end{aligned} \tag{14}$$

where $k_i = I_i/Ml$. Writing v for the body velocity $R^T\dot{x}$, the equations

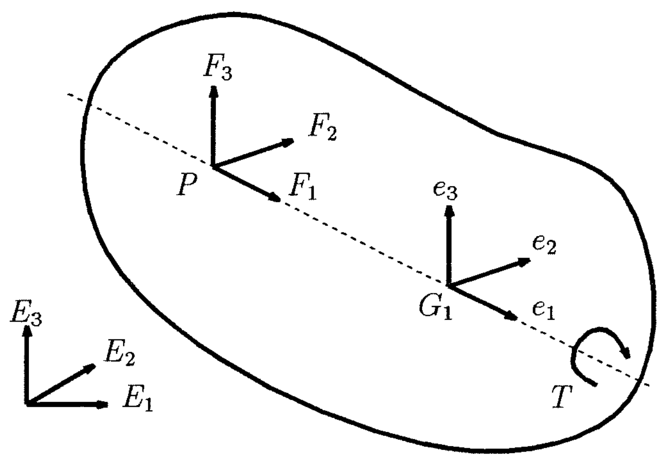

Figure 2: Rigid body in $\mathbb{R}^3$

correspond to the following Pfaffian system:

$$\begin{aligned} &\{R^T dR - \omega dt, R^T dx - v dt, \\ &(v_3\omega_1 - v_1\omega_3)dt - dv_2 - k_3 d\omega_3 - (k_2 - k_1)\omega_1\omega_2 dt, \\ &(-v_1\omega_2 - v_2\omega_1)dt + dv_3 - k_2 d\omega_2 - (k_1 - k_3)\omega_1\omega_3 dt\}. \end{aligned} \tag{15}$$

Now hypothesize that the system is flat with 3 of the flat outputs corresponding to a body fixed point and the fourth is a function of only the configuration and velocity variables (t, x, R, ω, v). The coordinates of a body fixed point are given by, $y = x + R\lambda$, where λ is the coordinate of the point with respect to (e_i) frame. Set $y = Y(t)$ for arbitrary $Y : \mathbb{R} \to \mathbb{R}^3$. Then we get $x = Y(t) - R\lambda, \quad v = R^T Y'(t) - \omega^\wedge \lambda$. Substituting these equations in the Pfaffian system (15), we get the reduced system I with 8 forms in coordinates (t, R, ω, v): a codimension 2 system. Derived system calculations are most conveniently carried out with the aid of a symbolic manipulator. The first derived system drops rank by 1 and $\{I, dt\}$ is integrable. Further calculations reveal that $\{I^{(1)}, dt\}$ is not integrable unless, one of the following equations held

1. $\lambda_1 = -k_2$,
2. $\lambda_1 = -k_3$,
3. $\lambda_2 = \lambda_3 = 0$,

4. $\lambda_2 = k_1 = 0$,
5. $\lambda_3 = k_1 = 0$,
6. $k_2 - k_3 = k_1 = 0$.

Due to symmetry (if we rotate body axis by 90^o about e_1 the equations remain unchanged), it suffices to consider cases 1, 3, 4 and 5. In all cases $\{I^{(1)}, dt\}$ is integrable and $I^{(2)}$ drops rank by 1, but $\{I^{(2)}, dt\}$ fails to be integrable and hence I cannot be flat. Hence appealing to Proposition 16 in previous section we conclude that coordinates of any body fixed point cannot provide a subset of the flat outputs.

5 Conclusion

In this paper we presented a method for testing flatness of a p-input system by reducing to the case of a single input. The method requires making an eduacated guess for $p-1$ of the flat outputs. We have shown in examples how physically meaningful outputs like body fixed points can provide succesful guesses.

Acknowledgement It is a pleasure to thank Richard Murray for suggesting the examples in this paper as well as for many valuable comments.

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A MapleV3 Worksheet for 3D Rigid Body

Three forces, one torque control system. Check for flatness. Assumption: three of the flat outputs depend on configuration variables only.

Loading a maple package for differential forms.

```
> with(fjeforms):
```

Declaring constants: $k1, \ldots, k3$ for the moments of inertia along the principal axes and $\lambda_1, \ldots, \lambda_3$ for the position of a body fixed point, which we take to be 3 of the four outputs.

```
> defform(k1=const,k2=const,k3=const);
> defform(lambda[1]=const,lambda[2]=const,lambda[3]=const);
> nforms(1,[Omega[1],Omega[2],Omega[3]]);
```

Introduce the skew symmetric matrix $Om = R^{-1}\mathrm{d}R$, then Om has constant coefficient structure equations.

```
> Om:=linalg[matrix]([[0,-Omega[3],Omega[2]],
> [Omega[3],0,-Omega[1]], [-Omega[2],Omega[1],0]]);
```

$$Om := \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

```
> R:=linalg[matrix](3,3):
> evalm(R &^ Om):
> drset:={seq(seq(d(R[i,j]) = "[i,j],i=1..3),j=1..3)};
```

$$\begin{aligned} drset := \Big\{ & \mathrm{d}\left(R_{2,2}\right) = -R_{2,1}\,\Omega_3 + R_{2,3}\,\Omega_1, \mathrm{d}\left(R_{3,2}\right) = -R_{3,1}\,\Omega_3 + R_{3,3}\,\Omega_1, \\ & \mathrm{d}\left(R_{1,3}\right) = R_{1,1}\,\Omega_2 - R_{1,2}\,\Omega_1, \mathrm{d}\left(R_{2,3}\right) = R_{2,1}\,\Omega_2 - R_{2,2}\,\Omega_1, \\ & \mathrm{d}\left(R_{3,3}\right) = R_{3,1}\,\Omega_2 - R_{3,2}\,\Omega_1, \mathrm{d}\left(R_{1,1}\right) = R_{1,2}\,\Omega_3 - R_{1,3}\,\Omega_2, \\ & \mathrm{d}\left(R_{2,1}\right) = R_{2,2}\,\Omega_3 - R_{2,3}\,\Omega_2, \mathrm{d}\left(R_{3,1}\right) = R_{3,2}\,\Omega_3 - R_{3,3}\,\Omega_2, \\ & \mathrm{d}\left(R_{1,2}\right) = -R_{1,1}\,\Omega_3 + R_{1,3}\,\Omega_1 \Big\} \end{aligned}$$

```
> Om &^ Om:
> dOm:=d(Om):
> dOmset:={dOm[3,2]=-""[3,2], dOm[1,3]=-""[1,3],
> dOm[2,1] = -""[2,1]};
```

$$\begin{aligned} dOmset := \Big\{ & \mathrm{d}\left(\Omega_2\right) = -\left(\Omega_3 \,\&\hat{}\, \Omega_1\right), \mathrm{d}\left(\Omega_3\right) = -\left(\Omega_1 \,\&\hat{}\, \Omega_2\right), \\ & \mathrm{d}\left(\Omega_1\right) = -\left(\Omega_2 \,\&\hat{}\, \Omega_3\right) \Big\} \end{aligned}$$

The functions $R_{i,j}$ are not independent, but, from the above, we can express their exterior derivatives in terms of the independent $\Omega_1, \ldots, \Omega_3$ and it turns out that we never encounter expressions of the $R_{i,j}$ of higher than first order, so we don't need to consider the algebraic relations that they satisfy here.

Read the system equations. We have a Pfaffian system with 5 generators in $\mathbb{R}^6 \times \mathbb{R}$ (thus codim =2)

```
> read(`system.txt`);
```

$$\eta_1 := \Omega_1 - \omega_1\, \mathrm{d}(t)$$

$$\eta_2 := \Omega_2 - \omega_2\, \mathrm{d}(t)$$

$$\eta_3 := \Omega_3 - \omega_3\, \mathrm{d}(t)$$

$$\begin{aligned}\eta_4 := {} & \mathrm{u1}(t)\,\mathrm{d}(R_{1,3}) + \mathrm{u2}(t)\,\mathrm{d}(R_{2,3}) + \mathrm{u3}(t)\,\mathrm{d}(R_{3,3}) - \lambda_1\,\mathrm{d}(\omega_2) \\ & + \left(R_{1,3}\left(\frac{\partial}{\partial t}\mathrm{u1}(t)\right) + R_{2,3}\left(\frac{\partial}{\partial t}\mathrm{u2}(t)\right) + R_{3,3}\left(\frac{\partial}{\partial t}\mathrm{u3}(t)\right)\right)\mathrm{d}(t) \\ & + \lambda_2\,\mathrm{d}(\omega_1) - \Big(\\ & \omega_2\,(R_{1,1}\,\mathrm{u1}(t) + R_{2,1}\,\mathrm{u2}(t) + R_{3,1}\,\mathrm{u3}(t) - \lambda_2\,\omega_3 + \lambda_3\,\omega_2) \\ & - \omega_1\,(R_{1,2}\,\mathrm{u1}(t) + R_{2,2}\,\mathrm{u2}(t) + R_{3,2}\,\mathrm{u3}(t) + \lambda_1\,\omega_3 - \lambda_3\,\omega_1)\Big)\,\mathrm{d}(t) \\ & - \mathit{k2}\,\mathrm{d}(\omega_2) + (\mathit{k3} - \mathit{k1})\,\omega_1\,\omega_3\,\mathrm{d}(t)\end{aligned}$$

$$\begin{aligned}\eta_5 := {} & \mathrm{u2}(t)\,\mathrm{d}(R_{2,2}) + \mathrm{u3}(t)\,\mathrm{d}(R_{3,2}) + \mathrm{u1}(t)\,\mathrm{d}(R_{1,2}) \\ & + \left(R_{1,2}\left(\frac{\partial}{\partial t}\mathrm{u1}(t)\right) + R_{2,2}\left(\frac{\partial}{\partial t}\mathrm{u2}(t)\right) + R_{3,2}\left(\frac{\partial}{\partial t}\mathrm{u3}(t)\right)\right)\mathrm{d}(t) \\ & - \lambda_3\,\mathrm{d}(\omega_1) + \lambda_1\,\mathrm{d}(\omega_3) - \Big(\\ & -\omega_3\,(R_{1,1}\,\mathrm{u1}(t) + R_{2,1}\,\mathrm{u2}(t) + R_{3,1}\,\mathrm{u3}(t) - \lambda_2\,\omega_3 + \lambda_3\,\omega_2) \\ & + \omega_1\,(R_{1,3}\,\mathrm{u1}(t) + R_{2,3}\,\mathrm{u2}(t) + R_{3,3}\,\mathrm{u3}(t) - \lambda_1\,\omega_2 + \lambda_2\,\omega_1)\Big)\,\mathrm{d}(t) \\ & + \mathit{k3}\,\mathrm{d}(\omega_3) - (\mathit{k1} - \mathit{k2})\,\omega_1\,\omega_2\,\mathrm{d}(t)\end{aligned}$$

```
> eta[4]:=simpform(subs(drset,eta[4])):
> eta[5]:=simpform(subs(drset,eta[5])):
```

Note that $\lambda_1 = -k2$ and $\lambda_1 = -k3$ are special cases, we will have to treat that as a separate case.

This is a small procedure that makes life easier.

```
> dd:=proc(ef)
```

```
>   local wstemp;
>   wstemp:=subs(drset,dOmset,d(ef));
>   wstemp:=simpform(wstemp)
> end:
```

Calculation of First Derived System.

Substitute *omset* to get an n-form modulo the system.

```
> omset:=simpform(solve({eta[1],eta[2],eta[3],eta[4],eta[5]},
>   {Omega[1],Omega[2],Omega[3],d(omega[2]),d(omega[3])}));
```

$$omset := \Big\{\Omega_1 = \omega_1\,\mathrm{d}(\,t\,), \Omega_3 = \omega_3\,\mathrm{d}(\,t\,), \Omega_2 = \omega_2\,\mathrm{d}(\,t\,),$$
$$\mathrm{d}\,(\omega_2) = \ldots, \mathrm{d}\,(\omega_3) = \ldots\}$$

```
> for k to 5 do deta[k]:=simpform(subs(omset,dd(eta[k]))) od;
```

$$deta_1 := -\,(\mathrm{d}\,(\omega_1)\ \&\hat{}\ \mathrm{d}(\,t\,))$$

$$deta_2 := -\frac{\lambda_2\ (\mathrm{d}\,(\omega_1)\ \&\hat{}\ \mathrm{d}(\,t\,))}{\lambda_1 + k2}$$

$$deta_3 := -\frac{\lambda_3\ (\mathrm{d}\,(\omega_1)\ \&\hat{}\ \mathrm{d}(\,t\,))}{\lambda_1 + k3}$$

$$deta_4 := \ldots$$

$$deta_5 := \ldots$$

Thus the first derived system is generated by the forms $sys_1, \ldots, sys_4$ given by:

```
> del:=coef(deta[1],d(omega[1]) &^ d(t));
```

$$del := -1$$

```
> sys[1]:=eta[2] - coef(deta[2],d(omega[1]) &^ d(t))/del*eta[1];
```

$$sys_1 := \Omega_2 - \omega_2\,\mathrm{d}(\,t\,) - \frac{\lambda_2\ (\Omega_1 - \omega_1\,\mathrm{d}(\,t\,))}{\lambda_1 + k2}$$

```
> sys[2]:=eta[3] - coef(deta[3],d(omega[1]) &^ d(t))/del*eta[1];
```

$$sys_2 := \Omega_3 - \omega_3\,\mathrm{d}(\,t\,) - \frac{\lambda_3\ (\Omega_1 - \omega_1\,\mathrm{d}(\,t\,))}{\lambda_1 + k3}$$

```
> AB:={A=coef(deta[4],d(omega[1]) &^ d(t))/del,
>      B=coef(deta[5],d(omega[1]) &^ d(t))/del}:
> sys[3]:=eta[4] - A*eta[1];
```

$$sys_3 := \ldots$$

```
> sys[4]:=eta[5] - B*eta[1];
```

$$sys_4 := \ldots$$

A and B are large expressions that we don't want to continue to expand, yet. Make its exterior differential an abstract expression in terms of the coframe $\Omega_1, \ldots, d\omega_1, \ldots, dt$.

```
> subs(AB,A):
> simpform(d("")):
> dra:=simpform(subs(drset, "")):
> dAset:={'d[1]'(A)=coef(dra,Omega[1]),
>    'd[2]'(A) =coef(dra,Omega[2]),
>    'd[3]'(A)=coef(dra,Omega[3]),
>    'd[4]'(A)=coef(dra,d(omega[1])),
>    'd[5]'(A)=coef(dra,d(omega[2])),
>    'd[6]'(A)=coef(dra,d(omega[3])),
>    'd[7]'(A) = coef(dra,d(t))};
```

$$dAset := \left\{ \mathrm{d}[2](A) = -\frac{\lambda_2 \, (R_{2,3}\,\mathrm{u2}(t) + R_{3,3}\,\mathrm{u3}(t) + R_{1,3}\,\mathrm{u1}(t))}{\lambda_1 + k2}, \ldots \right\}$$

```
> subs(AB,B):
> simpform(d("")):
> dra:=simpform(subs(drset, "")):
> dBset:={'d[1]'(B)=coef(dra,Omega[1]),
>    'd[2]'(B) =coef(dra,Omega[2]),
>    'd[3]'(B)=coef(dra,Omega[3]),
>    'd[4]'(B)=coef(dra,d(omega[1])),
>    'd[5]'(B)=coef(dra,d(omega[2])),
>    'd[6]'(B)=coef(dra,d(omega[3])),
>    'd[7]'(B) = coef(dra,d(t))};
```

$$dBset := \left\{ \mathrm{d}[4](B) = \frac{(\lambda_1 + k2 + k1)\,\lambda_2}{\lambda_1 + k2}, \ldots \right\}$$

```
> dABset:={d(A)=d(A,[Omega[1],'d[1]'],[Omega[2],'d[2]'],
>    [Omega[3],'d[3]'],[d(omega[1]),'d[4]'],
>    [d(omega[2]),'d[5]'],[d(omega[3]),'d[6]'],
>    [d(t),'d[7]']), d(B)=d(B,[Omega[1],'d[1]'],
>    [Omega[2],'d[2]'],[Omega[3],'d[3]'],
>    [d(omega[1]),'d[4]'],[d(omega[2]),'d[5]'],
>    [d(omega[3]),'d[6]'],[d(t),'d[7]']) };
```

$$
\begin{aligned}
dABset := \Big\{ & \mathrm{d}(B) = \mathrm{d}[1](B)\,\Omega_1 + \mathrm{d}[2](B)\,\Omega_2 + \mathrm{d}[3](B)\,\Omega_3 \\
& + \mathrm{d}[4](B)\,\mathrm{d}(\omega_1) + \mathrm{d}[5](B)\,\mathrm{d}(\omega_2) + \mathrm{d}[6](B)\,\mathrm{d}(\omega_3) \\
& + \mathrm{d}[7](B)\,\mathrm{d}(t), \mathrm{d}(A) = \mathrm{d}[1](A)\,\Omega_1 + \mathrm{d}[2](A)\,\Omega_2 + \mathrm{d}[3](A)\,\Omega_3 \\
& + \mathrm{d}[4](A)\,\mathrm{d}(\omega_1) + \mathrm{d}[5](A)\,\mathrm{d}(\omega_2) + \mathrm{d}[6](A)\,\mathrm{d}(\omega_3) \\
& + \mathrm{d}[7](A)\,\mathrm{d}(t) \Big\}
\end{aligned}
$$

Second Derived System

sol will give us the equations to substitute to calculate an n-form modulo the first derived system.

```
> sol:=simpform(solve({sys[1],sys[2],sys[3],sys[4]},
>   {Omega[2],Omega[3],d(omega[2]),d(omega[3])})):
> for k to 4 do dsys[k]:=simpform(dd(sys[k])):
>   dsys[k]:=subs(dABset,dsys[k]):
>   dsys[k]:=simpform(subs(drset,dOms et,dsys[k]));
>   dsys[k]:=simpform(subs(sol,dsys[k]))
> od:
> for k to 4 do dsys[k]:=simpform(dsys[k]) od:
```

These are long expressions, let's see which forms actually appear.

```
> for k to 2 do formpart(dsys[k]) od;
```

$$\mathrm{d}(t)\,\&\hat{}\,\Omega_1$$

$$\Omega_1\,\&\hat{}\,\mathrm{d}(t)$$

```
> for k from 3 to 4 do op(map(formpart,[op(dsys[k])])) od;
```

$$\mathrm{d}(\omega_1)\ \&\hat{}\ \mathrm{d}(t), \mathrm{d}(\omega_1)\ \&\hat{}\ \Omega_1, \mathrm{d}(t)\,\&\hat{}\,\Omega_1$$

$$\mathrm{d}(\omega_1)\ \&\hat{}\ \mathrm{d}(t), \mathrm{d}(\omega_1)\ \&\hat{}\ \Omega_1, \mathrm{d}(t)\,\&\hat{}\,\Omega_1$$

```
> for k from 3 to 4 do
>   eqn[k]:=coef(dsys[k],d(omega[1]) &^ Omega[1])
> od;
```

$$
\begin{aligned}
eqn_3 := - \Big(& \mathrm{d}[4](A)\,\lambda_1{}^2 + \mathrm{d}[4](A)\,\lambda_1\,k3 + \mathrm{d}[4](A)\,k2\,\lambda_1 + \mathrm{d}[4](A)\,k2\,k3 \\
& + \lambda_2\,\mathrm{d}[5](A)\,\lambda_1 + \lambda_2\,\mathrm{d}[5](A)\,k3 + \lambda_3\,\mathrm{d}[6](A)\,\lambda_1 + \lambda_3\,\mathrm{d}[6](A)\,k2 \Big) \\
& \Big/ ((\lambda_1 + k2)\ (\lambda_1 + k3))
\end{aligned}
$$

$$eqn_4 := \ldots$$

Need to substitute the exact expressions for A and B as well as for $d[i](A)$

etc ...

```
> subs(dAset,eqn[3]);
```

$$-\left(\frac{\%1\,\lambda_3\,\lambda_1^2}{\lambda_1+k3}+\frac{\%1\,\lambda_3\,\lambda_1\,k3}{\lambda_1+k3}+\frac{\%1\,\lambda_3\,k2\,\lambda_1}{\lambda_1+k3}+\frac{\%1\,\lambda_3\,k2\,k3}{\lambda_1+k3}\right.$$
$$-\frac{\lambda_2^2\,\lambda_3\,(k2-2\,k3-\lambda_1)\,\lambda_1}{(\lambda_1+k2)\,(\lambda_1+k3)}-\frac{\lambda_2^2\,\lambda_3\,(k2-2\,k3-\lambda_1)\,k3}{(\lambda_1+k2)\,(\lambda_1+k3)}$$
$$+\frac{\lambda_3\left(-\lambda_1\,k3-k2\,k3+k1\,\lambda_1+k1\,k2-\lambda_2^2-\lambda_1^2-k2\,\lambda_1\right)\lambda_1}{\lambda_1+k2}$$
$$\left.+\frac{\lambda_3\left(-\lambda_1\,k3-k2\,k3+k1\,\lambda_1+k1\,k2-\lambda_2^2-\lambda_1^2-k2\,\lambda_1\right)k2}{\lambda_1+k2}\right)\Big/\Big($$
$$(\lambda_1+k2)\,(\lambda_1+k3)\Big)$$
$$\%1 := k1+k3+\lambda_1$$

The terms that must vanish for FL are $cond_1,\ldots,cond_2$ (or possibly blow up, so that the previous calculations would have been no good).

```
> cond[1]:=factor(simplify("));
```

$$cond_1 := -2\,\frac{\left(\lambda_2^2\,k3+k2^2\,k1-\lambda_2^2\,k2+k1\,\lambda_1^2+2\,k1\,k2\,\lambda_1\right)\lambda_3}{(\lambda_1+k3)\,(\lambda_1+k2)^2}$$

```
> subs(dBset,eqn[4]):
> cond[2]:=factor(simplify("));
```

$$cond_2 := -2\,\frac{\left(\lambda_3^2\,k2+k1\,\lambda_1^2+k3^2\,k1+2\,k1\,\lambda_1\,k3-k3\,\lambda_3^2\right)\lambda_2}{(\lambda_1+k3)^2\,(\lambda_1+k2)}$$

Rearranging the numerators in $cond_1$ and $cond_2$, I make use of a simple personal procedure that factors only selected terms of an expression.

```
> read fac;
> op(2,cond[1]):
> fac(",{1,3});
```

$$k1\,(\lambda_1+k2)^2-(-k3+k2)\,\lambda_2^2$$

```
> op(2,cond[2]):
> fac(",{1,5});
```

$$k1\,(\lambda_1+k3)^2+(-k3+k2)\,\lambda_3^2$$

Discussion:

We want to find out if for some values of the λ_i, the system is flat. Necessary for this is that the terms $cond_1, \ldots, cond_2$ vanish. The cases where

$$\lambda[1] = -k2 \text{ or } \lambda[1] = -k3,$$

are seperate cases, that need to be dealt with. Aside from these, there are the following cases to consider (solving $cond_1 = cond_2 = 0$):

1. $\lambda_2 = \lambda_3 = 0$,
2. $\lambda_3 = 0, k1 = 0$,
3. $\lambda_2 = 0, k1 = 0$,
4. $k2 = k3, k1 = 0$.